PRAMANA
— journal of
physics

Vol. 61, No. 6
December 2003
pp. 1209–1214

# Non-linear high-frequency waves in the magnetosphere*

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MS received 1 April 2003; revised 19 August 2003; accepted 30 August 2003

**Abstract.** Using fluid theory, a set of equations is derived for non-linear high-frequency waves propagating oblique to an external magnetic field in a three-component plasma consisting of hot electrons, cold electrons and cold ions. For parameters typical of the Earth's magnetosphere, numerical solutions of the governing equations yield sinusoidal, sawtooth or bipolar wave-forms for the electric field.

**Keywords.** Magnetosphere; bipolar; sawtooth wave-forms.

**PACS Nos 52.35.Mw; 52.35.Qz; 90.30.Tz**

## 1. Introduction

Recent observations by the GEOTAIL [1], POLAR [2] and FAST [3] spacecrafts have indicated the presence of broadband electrostatic noise (BEN) in the Earth's magnetosphere. Several theoretical attempts have been made to explain the phenomena in terms of linear instabilities, especially for the high-frequency components [4]. The observed signatures are also found to display non-linear behaviour. In particular, BEN is found to have wave-forms of solitary bipolar electric field pulses which are called electrostatic solitary waves (ESW) [1].

Karovsky *et al* [5] have used a BGK analysis to theoretically describe the high-frequency ESW. Using counter-streaming electron and ion beams in a computer simulation experiment, Omura *et al* [6] have shown that the non-linear state of the system yields electric field structures similar to those of the ESW observed in BEN. More recently, Reddy *et al* [7] have used fluid theory to examine non-linear electric field structures in an electron–ion plasma. For sufficiently large values of the driver electric field, the governing equation for

*Article presented at the International Conference on the Frontiers of Plasma Physics and Technology, 9–14 December 2002, Bangalore, India.

the electric field, produced by a non-linear coupling between the ion acoustic wave and ion cyclotron oscillations, is found to admit both sawtooth and spiky periodic solutions which are in very good agreement with the observations of Ergun *et al* [3]. In this paper, we use the technique of Reddy *et al* [7] to investigate non-linear high-frequency electrostatic field structures for parameters characteristic of the Earth's magnetosphere. The basic theory is presented in the next section and the numerical results are presented in §3. Finally, our findings are summarized in §4.

## 2. Theory

We consider a three-component, collisionless, magnetized plasma consisting of hot electrons ($h$), cold electrons ($c$) and a cold ion species ($i$). The external magnetic field $\mathbf{B}_0$ is in the $x$–$z$ plane with $\theta = \theta(\mathbf{x}, \mathbf{B}_0)$. The continuity and momentum equations for the three species are given by, respectively,

$$\frac{\partial n_j}{\partial t} + \frac{\partial n_j v_{jx}}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v_{jx}}{\partial t} + v_{jx}\frac{\partial v_{jx}}{\partial x} + \frac{\alpha_j}{n_j m_j}\frac{\partial p_j}{\partial x} = -\frac{\varepsilon_j e}{m_j}\frac{\partial \phi}{\partial x} + \varepsilon_j \Omega_j v_{jy} \sin\theta, \tag{2}$$

$$\frac{\partial v_{jy}}{\partial t} + v_{jx}\frac{\partial v_{jy}}{\partial x} = \varepsilon_j \Omega_j v_{jz} \cos\theta - \varepsilon_j \Omega_j v_{jx} \sin\theta, \tag{3}$$

$$\frac{\partial v_{jz}}{\partial t} + v_{jx}\frac{\partial v_{jz}}{\partial x} = -\varepsilon_j \Omega_j v_{jy} \cos\theta, \tag{4}$$

where $\Omega_j = eB_0/m_j c$, $\varepsilon_j = +1(-1)$ for $j = i(c,h)$ and $\alpha_j = 1(0)$ for $j = h(c,i)$. The pressure balance equation for the hot electrons is given by

$$\frac{\partial p_h}{\partial t} + v_{hx}\frac{\partial p_h}{\partial x} + 3p_h\frac{\partial v_{hx}}{\partial x} = 0. \tag{5}$$

The system of equations is closed with the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{e}{\varepsilon_0}(n_i - n_c - n_h). \tag{6}$$

In the above equations, $n_j$ is the density of the $j$th species, $v_{jx}$, $v_{jy}$, $v_{jz}$ are the velocities of the $j$th species in the $x$-, $y$- and $z$-directions, respectively, and $\Omega_j$ is the electron (ion) cyclotron frequency for $j = e(i)$. It is noted that we allow spatial variation only in the $x$-direction.

Before proceeding to the non-linear analysis, we briefly discuss the linear modes of the system. Upon linearizing eqs (1)–(6), one obtains the dispersion relation

$$\omega^2 = \frac{\omega_{pi}^2\left(\omega^2 - \Omega_i^2 \cos^2\theta\right)}{\omega^2 - \Omega_i^2} + \frac{\omega_{pc}^2\left(\omega^2 - \Omega_e^2 \cos^2\theta\right)}{\omega^2 - \Omega_e^2} + \frac{\omega_{ph}^2\left(\omega^2 - \Omega_e^2 \cos^2\theta\right)}{\omega^2 - \Omega_e^2 - (3k^2 v_{th}^2/\omega^2)\left(\omega^2 - \Omega_e^2 \cos^2\theta\right)}. \tag{7}$$

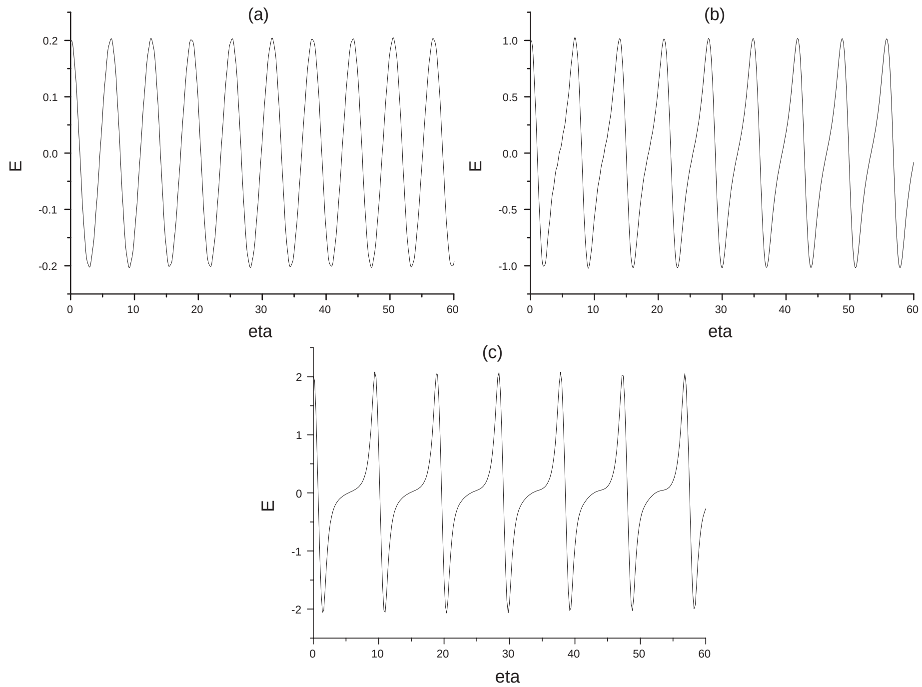


**Figure 1.** Numerical solution of the normalized electric field for the parameters $\theta = 2°$, $M = 2.5$, $n_{0c}/n_0 = 0.5$, $R = 10$, $\delta_i = \delta_h = \delta_c = 0$ and $E_0 = 0.2$ (**a**), 1.0 (**b**) and 2.0 (**c**).

For high-frequency waves one may neglect the ion motion. Then in the limit $\omega/k \ll v_{\rm th}$, where $v_{\rm th} = \sqrt{T_h/m_e}$ is the hot electron thermal velocity, (7) yields

$$\omega^2 = \frac{1}{2}(\omega_s^2 + \Omega_e^2)\left[1 \pm \sqrt{1 - \frac{4\omega_s^2\Omega_e^2\cos^2\theta}{(\omega_s^2+\Omega_e^2)^2}}\right], \tag{8}$$

where $\omega_s = \omega_{pc}/\sqrt{(1+1/k^2\lambda_{Dh}^2)}$, $\omega_{pc} = \sqrt{4\pi n_{0c}e^2/m_e}$ and $\lambda_{Dh} = \sqrt{T_h/4\pi n_{0h}e^2}$, where $n_{0c}(h_{0h})$ is the equilibrium density of the cold (hot) electrons with $n_{0c} + n_{0h} = n_0$. This dispersion relation has been derived by Mace and Hellberg [8] in the limit $\Omega_e < \omega_{pc}$, which is typical for many regions of the magnetosphere. They have shown that for $\omega_s \ll \Omega_e$, (8) has the following solutions:

$$\omega^+ \simeq \Omega_e + \frac{k^2 v_{se}^2}{2\Omega_e}\sin^2\theta, \quad \omega^- \simeq k v_{se}\cos\theta, \tag{9}$$

where $v_{se} = v_{\rm th}\sqrt{n_{0c}/n_{0h}}$ is the electron sound speed. The higher frequency solution $\omega^+$ is the electrostatic electron cyclotron wave (ECW), and the lower frequency solution $\omega^-$ is the electron acoustic wave (EAW).

It can be shown that the dispersion relation (9) is valid for those small wave numbers satisfying $k\lambda_{Dh} \ll (\omega_{pc}^2/\Omega_e^2 - 1)^{-1/2}$. For typical values of $R = \omega_{pe}/\Omega_e = 10$, where

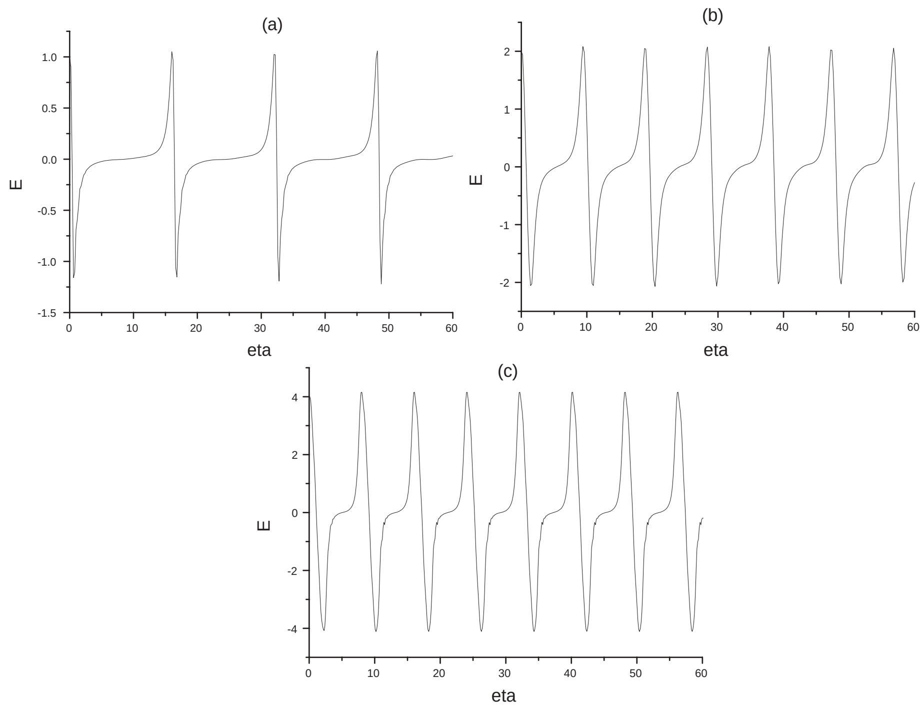


**Figure 2.** ESW for $n_{0c}/n_0 = 0.8$ (**a**), 0.5 (**b**) and 0.2 (**c**). Other parameters are fixed as in figure 1.

$\omega_{pe} = \sqrt{4\pi n_0 e^2/m_e}$ is the total electron plasma frequency, and $n_{0c}/n_0 = 0.5$, we find that the linear modes described by the dispersion relation (9) are possible for wavelenghts satisfying $k\lambda_{Dh} \ll 0.14$.

## 3. Non-linear numerical results

In the non-linear regime, the ECW and the EAW couple through the convective terms in eqs (2)–(5). In seeking traveling wave solutions we transform to a frame $\eta = (x - Vt)/(V/\Omega_e)$ in which we consider stationary solutions of the above set of equations. We normalize $v, t, x$ and $\phi$ with respect to $v_{\text{th}}$, $\Omega_{ce}^{-1}$, $\rho_e = v_{\text{th}}/\Omega_e$ and $T_h/e$. Upon defining $E = \partial\psi/\partial\eta$, $(\psi = e\phi/T_e)$, the above equations reduce to a non-linear set of first-order differential equations in which the continuity equation and the momentum equation in the $x$-direction are combined. In addition, $M = V/v_{\text{th}}$ as the Mach number and $\delta_j = v_{0j}/v_{\text{th}}$ is the normalized flow velocity of species $j$ at $\eta = 0$. The system of non-linear equations is then solved using a Runge–Kutta algorithm. The results are discussed below.

As found for the earlier study [7], the solution for the electric field is strongly dependent on the Mach number $M$ and the driving amplitude $E_0$ at $\eta = 0$. In figure 1 for fixed $M$, we vary $E_0$. Here $\theta = 2°$, so that the field is essentially parallel to $\mathbf{B}_0$. It is seen that as $E_0$ increases, the electric field evolves from a monochromatic wave to a sawtooth structure. For larger values of $E_0$, the wave becomes a periodic structure of bipolar pulses,

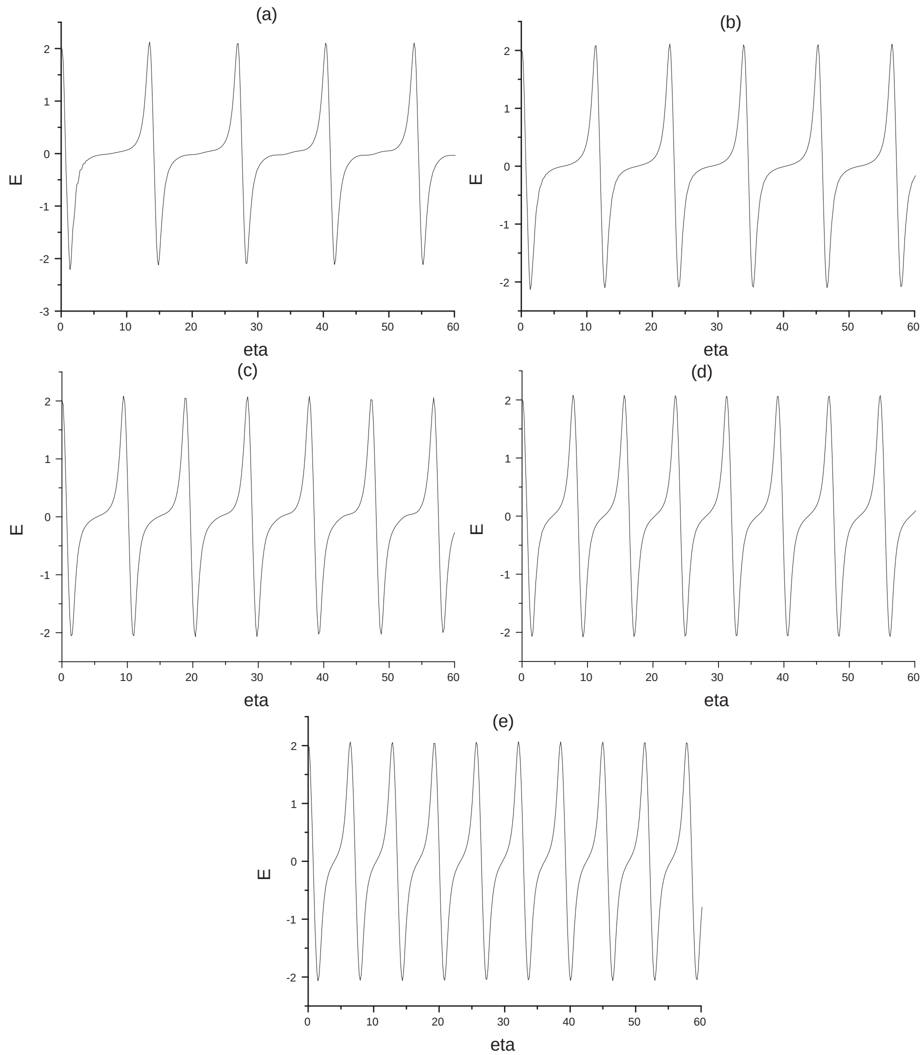


**Figure 3.** ESW for $\delta_h = -0.5$ (**a**), $-0.25$ (**b**), 0 (**c**), 0.25 (**d**) and 0.5 (**e**). Other parameters are fixed as in figure 1.

which are very similar to the ESW observed in the BEN in different regions of the magnetosphere. Typically for $\Delta x = 0$, $\Delta\eta = \mid \Omega_e \Delta t \mid$. Then for the linear wave ($E_0 = 0.2$) one finds $\omega = 1.0\Omega_e$, which is the ECW. As $E_0$ increases, in figure 1, the period of the non-linear structures increases. Consequently, the frequency decreases with $\omega = 0.89\Omega_e$ for $E_0 = 1.0$ and $\omega = 0.67\Omega_e$ for $E_0 = 2.0$. It is well-known [4] that a plasma with a cold- and a hot-electron component, with $0.2 < n_{0c}/n_{0h} < 0.8$, supports the high-frequency electron acoustic wave (EAW) which has a real frequency satisfying $\omega_{pi} < \omega < \omega_{pe}$.

Since $R = 10$ for the curves in figure 1, we identify the mode as the EAW satisfying the dispersion relation (9). Therefore, as the non-linearity increases in strength we have a transition from the higher-frequency ECW to the lower-frequency EAW. The bipolar electric field pulses (ESW) are, therefore, obtained for the latter mode. A variation of the relative densities of the cold and hot electrons (figure 2) shows that as $n_{0c}/n_0$ decreases from 0.8 to 0.2, the amplitude $E_0$ for the onset of the ESW increases from $E_0 = 1.0$ to 4.0.

The effect of a drifting hot-electron component is seen in figure 3. It is seen that for hot electron flow anti-parallel (parallel) to $\mathbf{B}_0$, increases (decreases) the period of the ESW. Similar studies for the cold electrons and the ions showed no effect on the period of the ESW.

## 4. Summary

In this study we have shown that a non-linear coupling between large-amplitude electron cyclotron and electron acoustic waves can account for the high-frequency component of the field-aligned bipolar electric field pulses observed within the broadband electrostatic noise in the auroral, polar and magnetotail regions of the Earth's magnetosphere. The free energy for these two modes could be either electron/ion drift or field aligned currents.

## References

[1] H Matsumoto, H Kojima, T Miyatake, Y Omura, M Okada, I Nagano and M Tsutsui, *Geophys. Res. Lett.* **21**, 2915 (1994)
[2] J R Franz, P M Kintner and J S Pichett, *Geophys. Res. Lett.* **25**, 1277 (1998)
[3] R E Ergun *et al*, *Geophys. Res. Lett.* **25**, 2041 (1998)
[4] D Schriver and M Ashour-Abdalla, *J. Geophys. Res.* **92**, 5807 (1987)
[5] V L Karovsky, H Matsumoto and Y Omura, *J. Geophys. Res.* **102**, 22, 131 (1997)
[6] Y Omura, H Kojima and H Matsumoto, *Geophys. Res. Lett.* **21**, 2923 (1994)
[7] R V Reddy, G S Lakhina, N Singh and R Bharuthram, *Nonlinear processes in geophysics* **9**, 25 (2002)
[8] R L Mace and M A Hellberg, *J. Geophys. Res.* **98**, 5881 (1993)