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Magnetic stabilization of transverse plasma instabilities†

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Waves, propagating transverse to the direction of the streaming of a plasma in the presence of a uniform external magnetic field, are unstable if the streaming exceeds a certain minimum value. The magnetic field reduces the growth rate of this instability, and also increases the value of the minimum streaming velocity, above which the system is unstable. The thermal motions in the plasma, however, tend to stabilize the system if the magnetic field is weak (i.e. $\Omega^2/k^2v_t^2 \lesssim 1$, Ω being the electron cyclotron frequency, k the characteristic wave-number, and v_t the thermal velocity); but, in case of strong magnetic field (i.e. $k^2v_t^2/\Omega^2 \ll 1$), they increase the growth rate, provided $c^2k^2/\omega_p^2 < \frac{1}{3}$ (ω_p being the electron plasma frequency).

1. Introduction

Plasmas streaming with a relative velocity exhibit the well-known two-stream (TS) instability (Bohm & Gross 1949; Büneman 1959; Jackson 1960; Kellog & Liemohn 1960; Buti 1963), whenever that velocity exceeds a critical amount. Büneman (1963) showed that, besides the TS instability, there exists a transverse instability for waves propagating perpendicular to the direction of relative streaming. According to him, this instability occurs for all streaming velocities. Momota (1966) gave the criterion for the existence of these transversal instabilities in cold as well as warm plasmas, and showed that the thermal effects were stabilizing. Lee (1969) extended Momota's work to include a uniform external magnetic field along the direction of the streaming; but, to avoid mathematical complications, he used fluid equations for the analysis, and concluded that both the magnetic field and the thermal effects were stabilizing. Strictly speaking, to get the correct quantitative behaviour of warm plasmas, we must use the Vlasov equation if collisions are negligible.† Here we have followed the latter approach, and we find that, for counter-streaming non-relativistic electron plasmas, the minimum streaming velocity required to produce the transverse instability in the absence of the magnetic field is (ckv_t/ω_p), rather than $v_t(1+c^2k^2/\omega_p^2)^{\frac{1}{2}}$ as derived by Lee (1969), or zero as concluded by Büneman (1963). The growth rate of the instability decreases with the increase in the magnetic field. The finite temperature of the plasma also decreases the growth rate if $k^2v_t^2 \gtrsim \Omega^2$, but has

† The referee has drawn our attention to Bornatici & Lee (1970), who use the Vlasov equation to describe the behaviour of an anisotropic-temperature plasma. They find that the perpendicular temperature $T_\perp$ is stabilizing, whereas the parallel temperature $T_\parallel$ is destabilizing, and that hence the conclusion of Lee (1969), that temperature is stabilizing, is valid only if $T_\perp \gg T_\parallel$. However, they do not explore the isotropic case; for this case, we find that the temperature can be destabilizing in some regions.

the opposite effect if $k^2v_t^2 \ll \Omega^2$ and $c^2k^2/\omega_p^2 < \frac{1}{3}$. Section 2 deals with the derivation of the general dispersion relation, and its analysis is given in § 3.

2. Dispersion relation

Let us consider two homogeneous streaming plasmas in which the ions only form the neutralizing background. Let the streaming $\mathbf{v}$ be along the direction of the uniform external magnetic field $\mathbf{B}_0$, which we take along the z-axis. For small perturbations in the system, the electrons are governed by the linearized Vlasov equation,

$$\frac{\partial f_1}{\partial t}+\mathbf{v}.\frac{\partial f_1}{\partial \mathbf{x}}-\frac{e}{m}\left(\mathbf{E}_1+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}_1\right).\frac{\partial f_0}{\partial \mathbf{v}}-\frac{e}{mc}(\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{v}}=0, \tag{1}$$

where f_0 is the equilibrium distribution function, and f_1, $\mathbf{E}_1$ and $\mathbf{B}_1$ are the perturbed distribution function, electric and magnetic fields, respectively. On taking the Fourier transforms in space, and Laplace transforms in time, of all the perturbed quantities, i.e.

$$f_1(\mathbf{x},\mathbf{v},t)=\int d^3\mathbf{k}\exp(i\mathbf{k}.\mathbf{x})\,F_k(\mathbf{k},\mathbf{v},\mathbf{t}),$$

and

$$\tilde{F}(\mathbf{k},\mathbf{v},s)=\int_0^\infty dt\exp(-st)\,F_k(\mathbf{k},\mathbf{v},t),$$

with $\operatorname{Re}s>0$, (1) can be simplified (Bernstein 1958; Buti 1963) to give

$$\tilde{F}_k(\mathbf{k},\mathbf{v},t)=\frac{1}{\Omega}\int_{-\infty}^{\phi}d\phi'\,G\left[F(\mathbf{k},\mathbf{v},0)+\mathrm{e}\left(\tilde{\varepsilon}_k+\frac{1}{c}\mathbf{v}'\times\tilde{\mathbf{B}}_k\right).\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}}\right]. \tag{2}$$

The integrating factor G in (2) stands for

$$\ln G=-\frac{1}{\Omega}\left[(s+ikv_z\cos\theta)(\phi-\phi')+ikv_\perp\sin\theta(\sin\phi-\sin\phi')\right], \tag{3}$$

and $\Omega=(eB_0)/(mc)$ is the cyclotron frequency. In writing (3), we have used cylindrical co-ordinates for $\mathbf{v}$, i.e.

$$V=(\mathbf{v}_\perp,\phi,v_z),\quad \mathbf{v}'=(v_\perp,\phi',v_z)\quad\text{and}\quad \mathbf{k}=(k\sin\theta,0,k\cos\theta).$$

On following the procedure outlined in Bernstein (1958) and Buti (1963), from (2), and from Fourier–Laplace-transformed Maxwell's equations, we get the relation,

$$\mathsf{R}.\tilde{E}_k=\mathbf{A}(\mathbf{k},\mathbf{s}), \tag{4}$$

where

$$\mathsf{R}=(c^2k^2+s^2)\,\mathsf{I}-c^2\mathbf{kk}-\sum_{\alpha=1}^{2}\frac{\omega_{p\alpha}^2}{N_\alpha}\int d^3\mathbf{vv}\int_{-\infty}^{\phi}d\phi'\,\frac{G}{\Omega}$$
$$\times\left[s\,\frac{\partial f_{0\alpha}}{\partial \mathbf{v}'}+i(\mathbf{k}.\mathbf{v}')\frac{\partial f_{0\alpha}}{\partial \mathbf{v}'}-i\mathbf{v}'\left(k\cdot\frac{\partial f_{0\alpha}}{\partial \mathbf{v}'}\right)\right], \tag{5}$$

and

$$\mathbf{A}=s\tilde{E}_k(t=0)+ic\mathbf{k}\times\mathbf{B}_k(t=0)+\sum_\alpha\int d^3\mathbf{vv}$$
$$\times\int_{-\infty}^{\phi}d\phi'\,\frac{G}{\Omega}\left[4\pi esF_{k\alpha}(\mathbf{v}',0)-\frac{\omega_{p\alpha}^2\mathbf{B}_k(t=0)}{N_\alpha c}\cdot\left(\mathbf{v}'\times\frac{\partial f_{0\alpha}}{\partial \mathbf{v}'}\right)\right] \tag{6}$$

represents simply the initial perturbations. The subscript α denotes the two electron streams. We may point out that (5) and (6) are a non-relativistic analogue of (16) and (17) of Buti (1963). If we consider only those perturbations for which A is analytic in the entire complex s-plane, then for the Laplace inversion of (4), we need to consider only the zeros of R, which are given by

$$|\mathsf{R}| = 0. \tag{7}$$

This is the required dispersion relation.

Now, for wave propagation perpendicular to the direction of $\mathbf{B}_0$, i.e. for $\theta = \frac{1}{2}\pi$, we notice that G is independent of v_z, and consequently, for the Maxwellian distribution

$$f_{0\alpha} = N_\alpha(2\pi v_{t\alpha}^2)^{-\frac{3}{2}} \exp[-(\mathbf{v} - u_\alpha \hat{e}_z)^2/2v_{t\alpha}^2], \tag{8}$$

the elements R_{xz}, R_{zx}, R_{yz} and R_{zy} vanish if the two plasmas are identical and are moving with same speed in opposite directions. For such contra-streaming plasmas, (7) reduces to give the two modes,

$$R_{zz} = 0 \quad \text{and} \quad (R_{xx}R_{yy} + R_{xy}R_{yx}) = 0.$$

The latter mode is independent of the streaming velocity, and was discussed by Bernstein (1958). So here we will consider only the ordinary wave characterized by $R_{zz} = 0$. According to (5), for transverse propagation, R_{zz} is given by

$$R_{zz} = s^2 + c^2k^2 + \sum_\alpha \frac{\omega_p^2}{N\Omega v_t^2}\int d^3\mathbf{v}\int_{-\infty}^{\phi} d\phi' G f_{0\alpha}(\mathbf{v}')[sv_z(v_z - u_\alpha) - iku_\alpha v_z v_\perp \cos\phi'], \tag{9}$$

with
$$G = \exp\left[-\frac{1}{\Omega}\{s(\phi - \phi') + ikv_\perp(\sin\phi - \sin\phi')\}\right]. \tag{10}$$

Following Bernstein (1958), (9) can be simplified to give

$$s^2 + c^2k^2 + \frac{\omega_p^2}{\Omega}\int_0^\infty dy \exp\left[-\frac{1}{\Omega}\{sy + \lambda\Omega(1-\cos y)\}\right].\left(s - \Omega\frac{\lambda u^2}{v_t^2}\sin y\right) = 0. \tag{11}$$

In writing (11), we have taken $u_1 = -u_2 = u$, $\frac{1}{2}N$ as the density of each stream, and $\lambda = k^2v_t^2/\Omega^2$. On further using the relation (Watson 1948),

$$\exp(\lambda\cos y) = \sum_{n=-\infty}^{\infty} I_n(\lambda)\exp(iny).$$

I_n being the Bessel function of first kind and of imaginary argument (putting $s = -i\omega$) equation (11) can be rewritten as

$$-\omega^2 + c^2k^2 + \omega_p^2 I_0(\lambda)\, e^{-\lambda}\left[1 - \lambda\frac{u^2}{v_t^2}.\frac{1}{1-\omega^2/\Omega^2}\right] + 2\omega_p^2\sum_{n=1}^{\infty} I_n(\lambda)\, e^{-\lambda}\left[\frac{(\omega/\Omega)^2}{(\omega/\Omega)^2 - n^2}\right.$$
$$\left.-\frac{\lambda u^2}{v_t^2}.\frac{1-n^2-\omega^2/\Omega^2}{\{1-([\omega/\Omega]-n)^2\}\{1-([\omega/\Omega]+n)^2\}}\right] = 0. \tag{12}$$

For cold plasma (i.e. $\lambda = 0$), (12) goes over to

$$\omega^2 - c^2k^2 - \omega_p^2[1 + k^2u^2/(\omega^2 - \Omega^2)] = 0, \tag{13}$$

which is identical to the equation derived by Momota (1966) and Lee (1969).

3. Marginal instability

The dispersion relation given by (12) can be analytically solved only in the limiting cases of strong ($k^2v_t^2/\Omega^2 \ll 1$) and weak ($k^2v_t^2/\Omega^2 \gg 1$) magnetic fields. However, for the general case, we can discuss the case of marginal instability, i.e. $\omega = 0$: in this case, (12) is replaced by

$$\frac{u_c^2}{v_t^2} = \frac{e^\lambda}{\lambda}\,[c^2k^2/\omega_p^2 + I_0(\lambda)\,e^{-\lambda}]\,.\left[I_0(\lambda) + \tfrac{1}{2}I_1(\lambda) - 2\sum_{n=2}^{\infty}\frac{I_n(\lambda)}{(n^2-1)}\right]^{-1}. \tag{14}$$

We may point out that (14) is obtained by taking the analytic continuation of (11) in the limit $\omega \to 0$ and $u_c = u(\omega = 0)$. Now u_c, which defines the boundary between the stable and the unstable regions, has a meaning only if the system under consideration has some instability. To ensure this, let us consider (12) in the limit $\lambda \ll 1$; to the lowest order in λ, this yields

$$\omega^6 - \omega^4(5\Omega^2 + c^2k^2 + \omega_p^2) + \omega^2\{4\Omega^4 + 5\Omega^2c^2k^2 + \Omega^2\omega_p^2(5-\lambda) - \omega_p^2k^2u^2\} + \chi = 0, \tag{15}$$

where

$$\chi = \Omega^2[(4-3\lambda)\,k^2u^2\omega_p^2 - 4\Omega^2\{c^2k^2 + \omega_p^2(1-\lambda)\}]. \tag{16}$$

Equation (15), being a cubic in ω^2, has at least one negative root (for ω^2) if $\chi > 0$; consequently, ω will be pure imaginary if $u > u^*$, where

$$u^* = \frac{\Omega}{k}\left[\frac{c^2k^2}{\omega_p^2}\left(1 + \frac{3}{4}\frac{k^2v_t^2}{\Omega^2}\right) + \left(1 - \frac{1}{4}\frac{k^2v_t^2}{\Omega^2}\right)\right]^{\frac{1}{2}}. \tag{17}$$

Note that, for $\lambda \ll 1$, u_c given by (14) is identical with u^*.

The variation of u_c with the magnetic field and the temperature is illustrated in figures 1 and 2. The magnetic field reduces the region of instability, but thermal effects are stabilizing only if $(ck/\omega_p)^2 > \frac{1}{3}$ for $\lambda < 1$; this restriction one gets immediately from (17). For $\lambda \gtrsim 1$, thermal effects are always stabilizing. For the given streaming velocity, the inequality $\chi > 0$, leads us to a bounded unstable region in k-space; the boundaries being defined by k_- and $k_+ (k_- < k_+)$, where $k_\pm^2$ are the roots of the biquadratic equation $\chi = 0$, i.e.

$$3k^4v_t^2u^2 + 4\frac{\Omega^2}{\omega_p^2}\{c^2\Omega^2 - (v_t^2 + u^2)\,\omega_p^2\}\,k^2 + 4\Omega^4 = 0. \tag{18}$$

The necessary criterion for k^2, of (18), to be real and positive is that

$$\Omega^2/\omega_p^2 < (v_t^2 + u^2)/c^2.$$

We may note that the value of u^* given by (17) for the case of a cold plasma agrees with that of Lee (1969) (cf. (15)), but, for a warm plasma, it differs from his results. The reason is that his value is derived from the dispersion relation (cf. (10) of Lee), which seems to be incorrect, as this does not take care of the well-known thermal correction for the transverse waves in the absence of streaming.

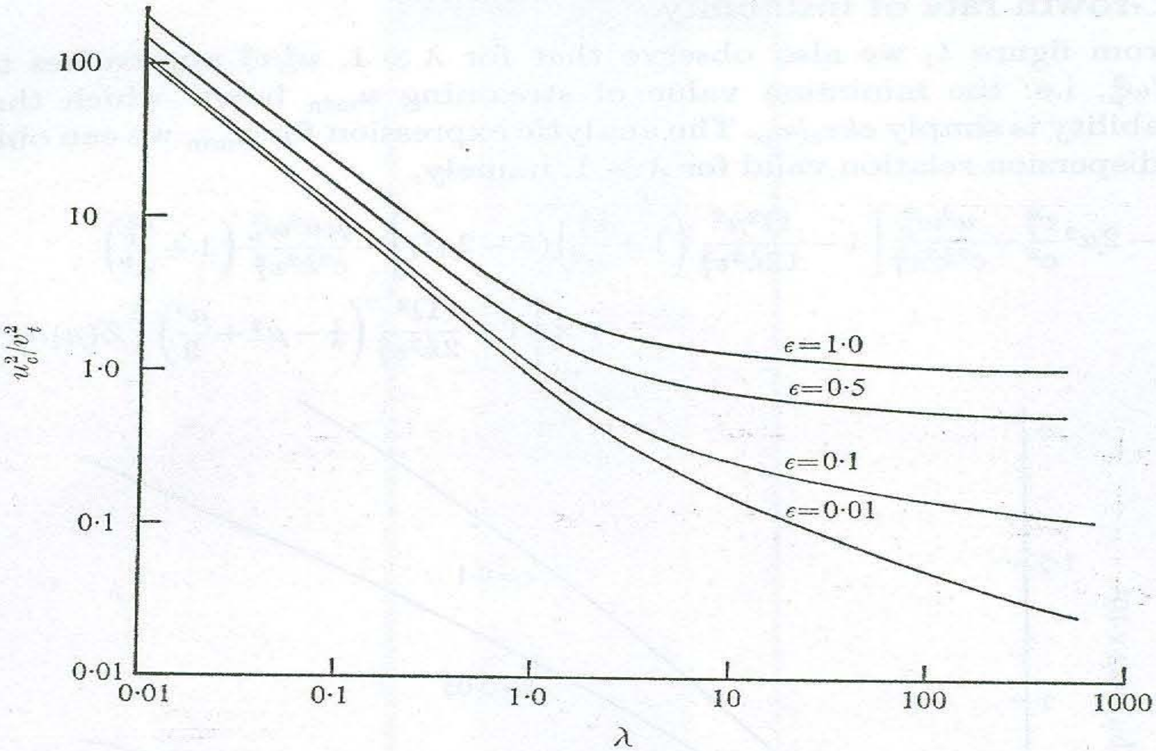

FIGURE 1. Variation of u_c^2/v_t^2 *vs*. $\lambda = k^2v_t^2/\Omega^2$ for $\epsilon = c^2k^2/\omega_p^2 = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}5$ and $1{\cdot}0$.

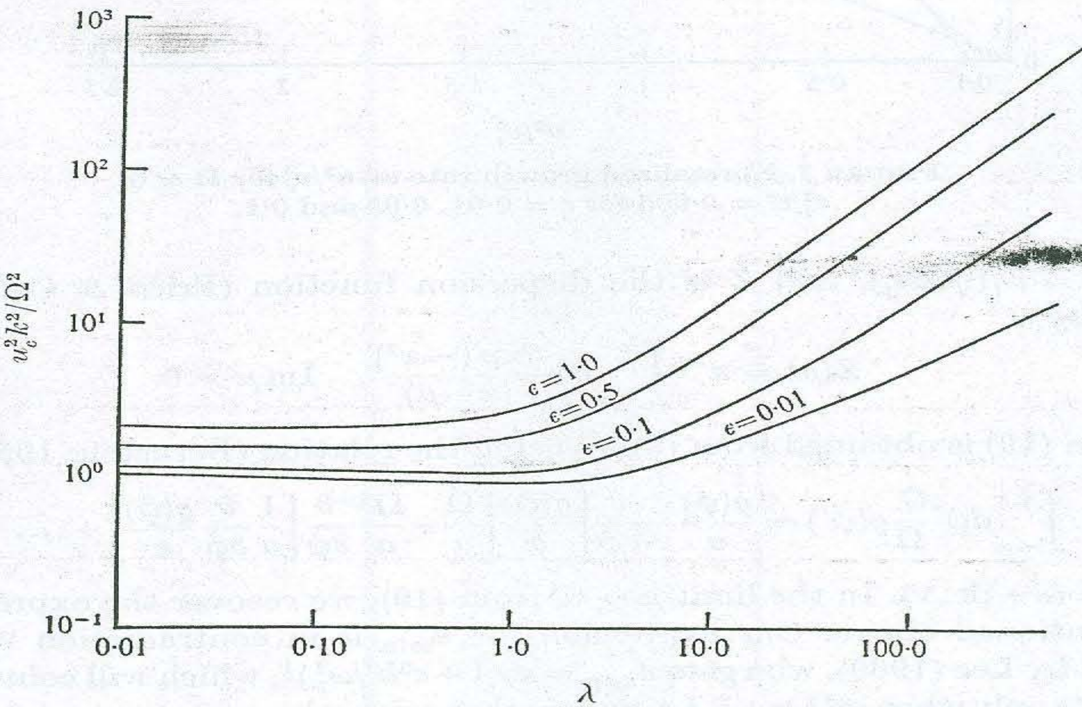

FIGURE 2. Variation of critical streaming with temperature for various values of $\epsilon = c^2k^2/\omega_p^2$.

4. Growth rate of instability

From figure 1, we also observe that for $\lambda \gg 1$, u_c^2/v_t^2 approaches the value c^2k^2/ω_p^2, i.e. the minimum value of streaming $u_{\min}$ below which there is no instability is simply ckv_t/ω_p. The analytic expression for $u_{\min}$ we can obtain from the dispersion relation valid for $\lambda \gg 1$, namely,

$$1-2\mu^2\frac{v_t^2}{c^2}-\frac{u^2\omega_p^2}{c^2k^2v_t^2}\left[1-\frac{\Omega^2\mu^2}{12k^2v_t^2}\left(1+\frac{v_t^2}{u^2}\right)(5-2\mu^2)\right]-\frac{\mu u^2\omega_p^2}{c^2k^2v_t^2}\left(1+\frac{v_t^2}{u^2}\right)\times\left[1+\frac{\Omega^2}{2k^2v_t^2}\left(\tfrac{1}{4}-\mu^2+\frac{\mu^4}{3}\right)\right]Z(\mu)=0, \quad (19)$$

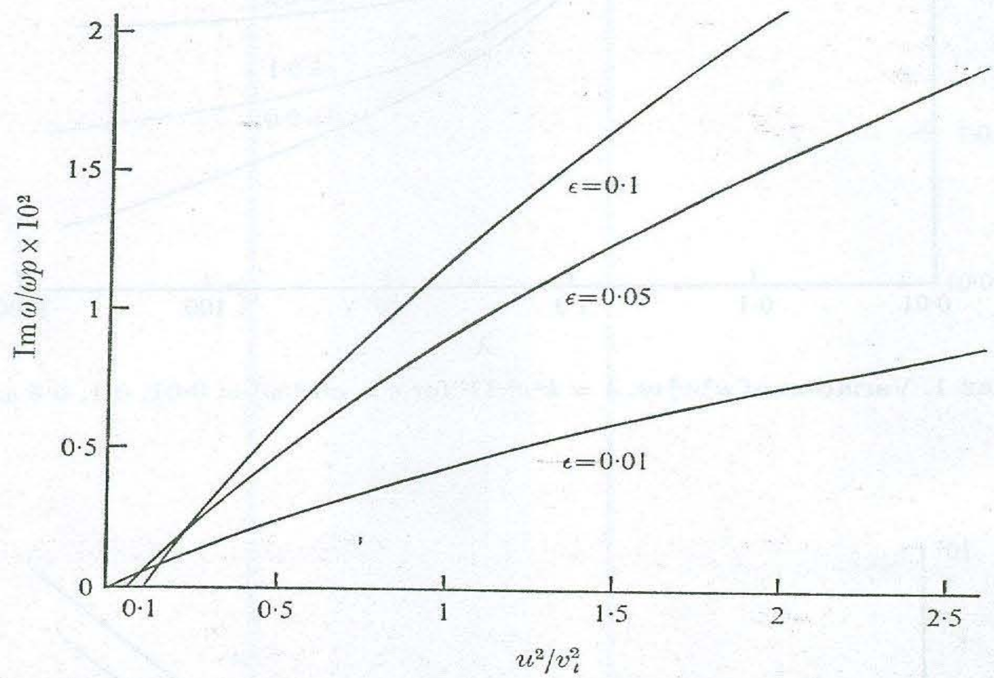

FIGURE 3. Normalized growth rate *vs.* u^2/v_t^2 for $\Omega = 0$, $v_t^2/c^2 = 0{\cdot}005$ for $\epsilon = 0{\cdot}01$, $0{\cdot}05$ and $0{\cdot}1$.

where $\mu = \omega/(\sqrt{2}kv_t)$, and Z is the dispersion function (Fried & Conte 1961) defined by

$$Z(\mu) = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} dx\,\frac{\exp(-x^2)}{(x-\mu)}, \quad \operatorname{Im}\mu > 0. \quad (20)$$

Equation (19) is obtained from (9) by using the relation (Bernstein 1958),

$$\int_{-\infty}^{\phi} d\phi'\,\frac{G}{\Omega}\,g(\phi') = \frac{g(\phi)}{a}-\frac{\partial}{\partial\phi}\left[\frac{g(\phi)}{a}\right]\frac{\Omega}{a}+\frac{\Omega^2}{a}\frac{\partial}{\partial\phi}\left[\frac{1}{a}\frac{\partial}{\partial\phi}\frac{g(\phi)}{a}\right]+\ldots, \quad (21)$$

with $a = (s+i\mathbf{k}.\mathbf{v})$. In the limit $\mu \to 0$, from (19), we recover the expression for $u_{\min}$ mentioned above. Our expression for $u_{\min}$ is in contradiction with that obtained by Lee (1969), who gets $u_{\min} = v_t(1+c^2k^2/\omega_p^2)^{\frac{1}{2}}$, which will coincide with our results only when $c^2k^2 \gg \omega_p^2$, i.e. only in the region where transverse instabilities are unimportant compared with the longitudinal TS instabilities (Büneman

1963). This contradiction is probably due to the discrepancy in Lee's dispersion relation, mentioned earlier.

For $|\mu| \gg 1$, (19) reduces to

$$\omega^2 = c^2k^2 + \omega_p^2 + \frac{5}{8}\frac{\omega_p^2}{c^2k^2+\omega_p^2}\left(\frac{\Omega^2}{k^2v_t^2}\right)\left(1+\frac{u^2}{v_t^2}\right)+\dots, \tag{22}$$

which shows immediately that there is no instability. For other values of μ, (19) is solved numerically; the growth rate of instability thus obtained is shown in figures 3 and 4. It decreases with an increase in the magnetic field.

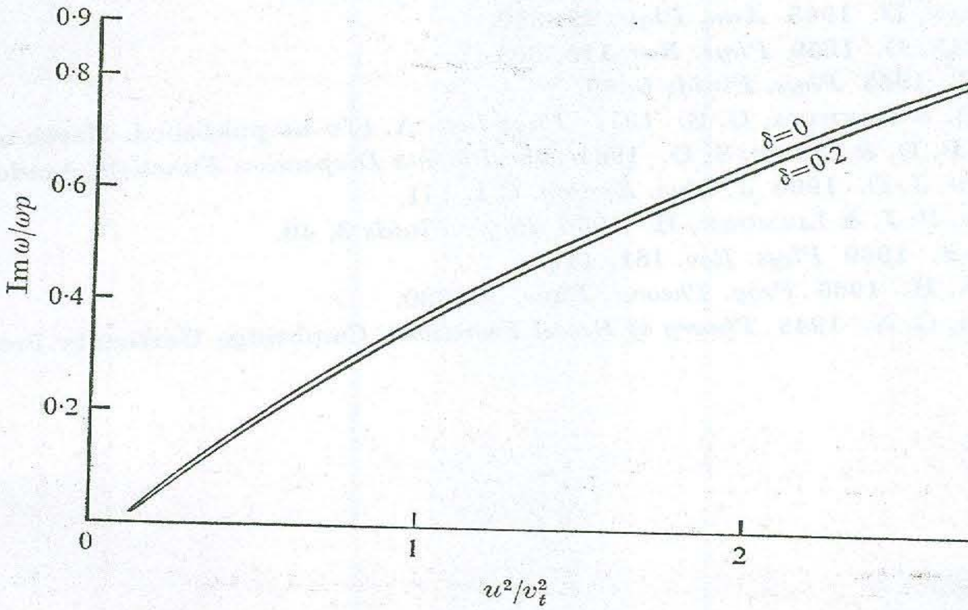

FIGURE 4. Normalized growth rate *vs.* u^2/v_t^2 for $\epsilon = 0{\cdot}1$, $v_t^2/c^2 = 0{\cdot}005$ and for $\delta = \Omega^2/k^2v_t^2 = 0$ and $0{\cdot}2$.

For $\Omega = 0$, (19) goes over to Büneman's (1963) dispersion relation. However, Büneman concludes that, in the absence of a magnetic field, instability occurs for all values of u, i.e. $u_{\min} = 0$. But it is not at all clear from his paper how he draws this conclusion, which is in contradiction with our conclusion that $u_{\min}$ is finite.

5. Conclusions

For identical contra-streaming plasmas in the presence of uniform magnetic field along the direction of streaming the ordinary mode gets modified by the streaming, whereas the extraordinary mode remains unaffected. The transverse instability exists only if the streaming exceeds a certain minimum value. The magnetic field has a stabilizing effect on this instability, but the temperature is destabilizing if the field is strong and $c^2k^2/\omega_p^2 < \frac{1}{3}$, otherwise thermal effects are also stabilizing.

The ion motion, which has been neglected in the present study, may make an important contribution; this is currently under investigation, and will be reported in Buti & Lakhina (1971).

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